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A Method of Optimal Investment with Stage-by-Stage Conditional Value at Risk (CVaR) Constraints and Known Parameters of Return Vectors

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Abstract—The paper investigates a multi-stage investment problem under Conditional Value at Risk (CVaR) constraints with: a given security level for bankruptcy, short selling permission, a normal and an elliptical total return distribution models. The purpose of the work is to find a method of determining the optimal investment in this problem at each stage. As a result of the study, an optimal investment strategy is found and it is shown that the optimal investment portfolio at each stage does not depend on the value of the investor's capital, but depends only on the number of stage. It is shown that the multi-stage problem can be reduced to a finite number of one-stage optimization problems, which are problems of conic programming. For the one-stage problem, conditions for the non-emptiness of the set of admissible portfolios are given and the Kuhn–Tucker theorem is applied. Additionally, this paper presents a numerical example of finding the optimal investment based on the open data on rates of assert prices of companies on the stock exchange.

Keywords: optimal investment portfolios, risk measure, probability constraint CVaR, bankruptcy

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1. INTRODUCTION

The formalization of the concept of "optimal investment portfolio" and methods for its finding were first proposed by Markowitz in 1952 in [1], where a mathematical model for the determination of an optimal investment portfolio was described. Markowitz used the variance of returns as a risk measure that allows one to obtain an estimate of the financial risk for a portfolio of assets. After using variance as a risk measure, many scientists have explored alternative risk measures [2, 3]. One such risk measure is the Value at Risk (VaR), which has been widely used since the 1980s. VaR is an estimate of the amount that expected losses will not exceed with a given probability, expressed in monetary terms. In the [4] quantiles, i.e., VaR constraints, were used in the formulation and solutions of optimization problems. In [5], variance as a risk measure, which Markowitz used in his work, is compared with the risk measure VaR. The authors conclude that, in general, the risk measure VaR has advantages, but is not free from disadvantages; for example, using VaR it is impossible to estimate the size of losses outside the confidence level. Some researchers [6] have pointed out that VaR is not a coherent measure of risk (see the definition, for example, in [6]), since it does not satisfy the subadditivity property except in the case of a normal risk distribution. On the other hand, [5, 7] noted that the risk measure VaR is widely used by regulators in the financial industry.

The risk measure CVaR introduced in the literature in recent decades is a coherent risk measure; in [6], the authors compared the risk measure VaR and CVaR and came to the conclusion that in the absence of a risk-free asset in the investment portfolio, the risk measure CVaR is more effective than the risk measure VaR. In addition, the authors note that CVaR takes into account outliers, which is critical for valuing risky and volatile assets. In [8], the authors conclude that despite the advantage of the risk measure CVaR over the risk measure VaR in the general case, it makes sense to take into account both of these measures to estimate the risk of an investment portfolio.

In [9], a multi-stage investment problem with VaR constraints but without the possibility of bankruptcy was investigated. On the other hand, (see, for example, [10]) bankruptcy, i.e., a decrease in the investor's capital below a given value, when further investment transactions are prohibited for the remaining time interval, plays a significant role in the estimation of financial strategy. An optimal investment strategy with stage-by-stage quantile (i.e., VaR) constraints was found in [11], where the possibility of bankruptcy is assumed.

In the presented work, a multi-stage investment problem with the possibility of bankruptcy is studied, where, unlike [11], stage-by-stage CVaR constraints are imposed; the advantages of this approach over the introduction of VaR constraints are described above. It is shown that the considered multi-stage risk sharing problem has a solution in which each optimal portfolio at stage t depends only on the stage number t and does not depend on the value of the investor's current capital ($x > 0$). It is proved that the original multi-stage problem is reduced to solving a finite number of one-stage cone optimization problems, where the objective functions are determined by a recurrent formula, which makes it relatively easy to find the optimal investment strategy. Unlike [12, 13], where the risk of bankruptcy at each stage was only estimated by the use of Chebyshev's inequality, here the authors investigate the problem by solving dynamic programming equations that take into account the possibility of bankruptcy.

Section 2 considers a one-stage problem with CVaR constraint. Necessary and sufficient conditions for the set of admissible portfolios to be non-empty and for the Slater regularity condition to be satisfied are given, and the Kuhn–Tucker theorem is used to determine the solution to this cone optimization problem. Section 3 is devoted to the study of optimization of investment strategy in a multi-stage problem with CVaR constraints. It is shown that each component of the optimal portfolio depends only on the investment stage number and does not depend on the investor's current capital; the necessary conditions for the optimality of the strategy are found. A model different from the previously considered normal model is investigated in Section 4, where a generalization of the normal model to elliptical distributions is studied. In Section 5, a numerical example is solved, illustrating, based on real data, the finding of an optimal strategy for investing in three large companies. Section 6 contains concluding remarks.

2. ONE-STAGE OPTIMAL PORTFOLIO CONSTRUCTION PROBLEM

Let us first study a one-stage model for choosing an optimal investment portfolio (see, for example, [6, 11]), where the random vector of asset returns $R = (R_0, \ldots, R_n)$, and R_i represents the change in the value of the ith asset from the current value as a percentage. In terms of stock prices, this means that $R_i = p_1/p_0$, where p_0 is the current price of the ith asset (a deterministic variable), p_1 is the price of the ith asset at the next quotation (a random variable). Let $R_0 = m_0$ almost surely (a.s.), i.e., it is a risk-free asset. Let $\overline{a} \in R^{n+1}$ is an investment portfolio where a_i is the percentage of the initial capital $x_0 > 0$ invested in the ith asset. The budget constraint $\sum_{i=0}^{n} a_i = 1$ means the investor's self-financing (there is no inflow of funds from outside, and the investor's available funds are invested only in the assets of this market) and, at the same time, permission for "short sales," i.e., the possibility of borrowing some assets at their current value with the purpose of investing this money in other assets. The function to be maximized is the mathematical expectation of the final investor's capital

$$EX_{\overline{a}} = Ex_0 \sum_{i=0}^{n} a_i R_i = x_0 \sum_{i=0}^{n} a_i m_i,$$

where $m_i = ER_i$ and $x_0 > 0$ is the initial capital.

To formulate the CVaR risk measure (see [6]) in the problem under investigation, we first need to define the risk measure Value at Risk (VaR) for random income Z:

$$\mathrm{VaR}_\alpha(Z) = -z_\alpha = -\inf\{t : P(Z < t) \geqslant \alpha\},$$

where $\alpha \in (0, 1/2)$ is a given significance level. Conditional Value at Risk (CVaR) is a risk measure that has the meaning of expected losses in the case of exceeding the conditional risk measure VaR with a given significance level α:

$$\mathrm{CVaR}_\alpha(Z) = E\{Z|\, Z \geqslant \mathrm{VaR}_\alpha(Z)\}.$$

Let $f_Z(z)$ denote the density function of the standardized income distribution $(Z - EZ)/\sqrt{DZ}$. Then [6]

$$\mathrm{CVaR}_\alpha(Z) = -EZ + \sqrt{DZ}k,$$

where

$$k = \frac{-\int\limits_{-\infty}^{-z_\alpha} z f_Z(z) dz}{\alpha}.$$

Assuming a normal approximation of the investor's final capital $X_a = x_0 \sum_{i=0}^{n} a_i R_i$, which is widely used in portfolio theory (see, for example, [6, 9]), we obtain an expression for the parameter k (see its definition above). Since $\phi'(x) = -x\phi(x)$, where $\phi(x)$ is the density of the standard normal distribution, it is easy to see that $k = \phi\left(\Phi^{-1}(\alpha)\right)/\alpha$. Then

$$\mathrm{CVaR}_\alpha(X_a) = -EX_a + \sqrt{DX_a}\frac{\varphi(\Phi^{-1}(\alpha))}{\alpha},$$

where $\Phi(x)$ is the standard normal distribution function, α is the specified significance level.

Let us introduce the constraint CVaR for the problem under consideration:

$$\mathrm{CVaR}_\alpha(X_a) \equiv -(x_0\langle a, \Delta m\rangle + x_0 m_0) + x_0\sqrt{aCa'}\frac{\phi\left(\Phi^{-1}(\alpha)\right)}{\alpha} \leqslant dx_0.$$

Here Δm denotes the vector $(m_1 - m_0, \ldots, m_n - m_0)$, $\langle a, \Delta m\rangle$ is the scalar product $\sum_{i=1}^{n} a_i \Delta m_i$, d is the share in percent of the initial capital $x_0 > 0$, C is the covariance matrix of $n \times n$ risky assets.

Summarizing all the above considerations, we formulate a one-stage optimization problem:

$$\begin{cases} EX_a \equiv \langle a, \Delta m\rangle + m_0 \to \max, \\ a \in D = \left\{ a \in R^n : \sqrt{aCa'} \leqslant \dfrac{\langle a, \Delta m\rangle + m_0 + d}{\varphi(\Phi^{-1}(\alpha))/\alpha} \right\}. \end{cases} \tag{1}$$

Below we will use the following natural assumptions: $0 < m_0 < \min_{i=1,\ldots,n} m_i$, the covariance matrix of risky assets C is positive definite.

Further in this section, we simply present modifications of the statements obtained earlier (see [11, 14]) for the VaR constraints and for the case of CVaR constraints (see (1)), which will be used in solving a multi-stage problem.

By definition, a second-order cone (see, e.g., [15]) is $K = \{(a,t) \in R^{n+1} : \sqrt{aCa'} \leqslant t\}$. It is known that such a cone is regular, in other words, it is convex, closed, $IntK \neq \emptyset$ and, if $x \in K$, $-x \in K$, then $x = 0$. Problem (1) can be rewritten as a cone programming problem [15]

$$\max\langle a, \Delta m\rangle \text{ under constraints } \left(a, \frac{\langle a, \Delta m\rangle + m_0 + d}{\frac{\varphi(\Phi^{-1}(\alpha))}{\alpha}}\right) \in K. \tag{2}$$

To solve (2), we need a description of the cone dual to K. By definition, the dual cone is $K^* = \{x \in R^{n+1} : \langle x, y\rangle \geqslant 0 \text{ for all } y \in K\}$. In problem (2), it is defined [3] as

Lemma 1. *The dual cone is equal to*

$$K^* = \left\{(u,v) \in R^{n+1} : \sqrt{uC^{-1}u'} \leqslant v\right\}.$$

Below we present a condition sufficient for the fulfillment of the Slater condition (see the definition, for example, in [15]) in problem (2). Let j denote the index at which the function $\sigma_i \frac{\varphi(\Phi^{-1}(\alpha))}{\alpha} - \Delta m_i$ reaches its minimum, $i = 1, \ldots n$, where $\sigma_i = \sqrt{DR_i}$ is the standard deviation.

Statement 1. *If*

$$\sigma_j \frac{\varphi(\Phi^{-1}(\alpha))}{\alpha} - \Delta m_j < 0, \tag{3}$$

then the interior $IntD \neq \emptyset$, i.e., the Slater condition in problem (1) is met and, therefore, in problem (2) also.

Proof. Let a^j be an investment portfolio of the form $(0, \ldots, 0, a_j, 0, \ldots 0)$, where a_j is in the jth place, and all other components of the investment portfolio are zeros. Note that in the case under consideration short sales are allowed, i.e., a_j can be less than zero. CVaR the constraint in the one-stage problem (1) takes the form

$$|a_j|\,\sigma_j \frac{\varphi(\Phi^{-1}(\alpha))}{\alpha} - a_j \Delta m_j \leqslant d + m_0.$$

Let us consider the minimum of the left-hand side of this expression:

$$\rho = \min_a \left\{|a_j|\,\sigma_j \frac{\varphi\left(\Phi^{-1}\left(\alpha\right)\right)}{\alpha} - a_j \Delta m_j\right\}.$$

Show that

$$\rho = -\infty.$$

Indeed, consider the case $a_j \geqslant 0$. The left-hand side of the CVaR constraint is rewritten as follows:

$$a_j \sigma_j \frac{\varphi\left(\Phi^{-1}\left(\alpha\right)\right)}{\alpha} - a_j \Delta m_j = a_j \left(\sigma_j \frac{\varphi\left(\Phi^{-1}\left(\alpha\right)\right)}{\alpha} - \Delta m_j\right).$$

Since by assumption $\sigma_j \frac{\varphi\left(\Phi^{-1}(\alpha)\right)}{\alpha} - \Delta m_j < 0$, we have

$$\min_{a_j} \left\{a_j \left(\sigma_j \frac{\varphi\left(\Phi^{-1}\left(\alpha\right)\right)}{\alpha} - \Delta m_j\right)\right\} = -\infty.$$

Summarizing the above reasoning, we obtain: if $\sigma_j \frac{\varphi(\Phi^{-1}(\alpha))}{\alpha} - \Delta m_j < 0$, then $IntD \neq \emptyset$. Statement 1 is proved.

Note that from the proof of Statement 1 it obviously follows (see (3)) a sufficient condition for the non-emptiness of an admissible set D, $\sigma_j \frac{\varphi(\Phi^{-1}(\alpha))}{\alpha} - \Delta m_j \leqslant 0$.

Statement 2. *If condition* (3) *is satisfied, then the admissible portfolio* a^* *is optimal in problem* (2) *if and only if there exists a vector* $(\lambda_1, \ldots, \lambda_n, \mu) \in K^*$ *such that*

$$\Delta m \left(1 + \frac{\mu}{\frac{\varphi(\Phi^{-1}(\alpha))}{\alpha}}\right) + \lambda = 0$$

and

$$a^* \lambda' + \frac{\mu(\langle a^*, \Delta m\rangle + m_0 + d)}{\frac{\varphi(\Phi^{-1}(\alpha))}{\alpha}} = 0, \text{ where } \lambda = (\lambda_1, \ldots, \lambda_n).$$

This statement is an obvious consequence of the Kuhn–Tucker theorem [15] for the concave cone programming problem (2). Note that without the concavity condition of the objective function in (2), which is now linear, the statement gives only necessary optimality conditions.

3. MULTI-STAGE OPTIMAL PORTFOLIO CONSTRUCTION PROBLEM

In a multi-stage problem, the investment horizon is divided into T parts or time moments $0, 1, \ldots, T$. The random vector of asset returns for time moment t is denoted as $R^t = (R_0^t, \ldots, R_n^t)$, where $R_0^t = m_0^t$ a.s. is the return of the risk-free asset. The vectors R^t are assumed to be independent, as in the works on investment optimization in multi-stage models with bankruptcy [7, 12]. We denote the mathematical expectation of the return on the ith asset by $m_i^t = ER_i^t$. The covariance matrices of risky asset returns C_t are positive definite at each stage t.

The term "investor bankruptcy" means the following. If the investor's current capital X_t falls below the threshold $r = 0$, then the investor cannot make any transactions with assets (buy, sell, borrow) and the value of the capital X_t is fixed from the moment t until the last time moment T. After applying the normal approximation to the percentage increase in capital over the interval $[t, t+1]$, we obtain a normally distributed random variable $Y_a^t := \sum_1^n a_i(R_i^t - m_0^t) + m_0^t$ with parameters $\mu^t(a) = \langle a, \Delta m^t\rangle + m_0^t$ and $\sigma^t(a) = \sqrt{aC_t a'}$. Then, the stage-by-stage CVaR constraints (see Section 2) have the form

$$-X_t\langle a, \Delta m^t\rangle + X_t m_0^t + X_t\sqrt{aC_t a^T}\frac{\varphi\left(\Phi^{-1}\left(\alpha^t\right)\right)}{\alpha^t} \leqslant d^t X_t \text{ for all } X_t = x > 0.$$

Here $\Delta m^t = (m_1^t - m_0^t, \ldots, m_n^t - m_0^t)$, $d^t > 0$ is the share in percentage of the current capital, limiting the risk measure CVaR; $\alpha^t \in (0, 1)$ is the significance level, $t = 0, \ldots, T-1$. We denote the investment strategy by $A = (a^0, \ldots, a^{T-1})$, recall that short sales are allowed.

Then, the equation of capital dynamics for the chosen investment strategy is

$$X_{t+1} = \begin{cases} X_t\left[\sum\limits_{i=1}^n a_i^t\left(R_i^t - m_0^t\right) + m_0^t\right], & \text{when } X_t > 0, \\ X_t, & \text{when } X_t \leqslant 0, \\ t = 0, \ldots, T-1; X_0 = x_0, & \text{when } x_0 > 0. \end{cases} \tag{4}$$

It is assumed that the investor's goal is to maximize the average value of the final capital. Thus, the problem under consideration is the problem of optimal control of a Markov chain in the presence of a set of absorbing states $\{x : x \leqslant 0\}$:

$$\begin{cases} J[A] \equiv E(X_T) \to \max, \quad A \in \mathbf{A} \text{ under constraints (4) and} \\ -(\langle a, \Delta m^t\rangle + m_0^t) + \sqrt{aC_t a^T}\dfrac{\varphi\left(\Phi^{-1}\left(\alpha^t\right)\right)}{\alpha^t} \leqslant d^t, \end{cases} \tag{5}$$

where $\mathbf{A}$ is the set of all investment strategies that are predictable in the sense of natural filtering.

Define the Bellman functions (value functions) as $V_t(x) = \max_A EX_T$ for the controlled process on the interval $[t, T]$ with the initial state $X_t = x$. Then $V_T(x) = x$;

$$
\begin{aligned}
V_{T-1}(x) &= \max_{a \in D_{T-1}} xEY_a^{T-1} = \max_{a \in D_{T-1}} xG_{T-1}(a) = xG_{T-1}\left(a_*^{T-1}\right), \text{ if } x > 0, \\
V_{T-1}(x) &= x, \text{ if } x \leqslant 0; \\
V_{T-2}(x) &= \max_{a \in D_{T-2}} x\left\{E\left[G_{T-1}\left(a_*^{T-1}\right)Y_a^{T-2} | Y_a^{T-2} > 0\right]\right. \\
&\quad \left. \times P\left(Y_a^{T-2} > 0\right) + E\left[Y_a^{T-2} | Y_a^{T-2} \leqslant 0\right] P\left(Y_a^{T-2} \leqslant 0\right)\right\} \\
&= \max_{a \in D_{T-2}} xG_{T-2}(a) \\
&= xG_{T-2}\left(a_*^{T-2}\right) \text{ when } x > 0 \text{ and } V_{T-2}(x) = x \text{ when } x \leqslant 0; \\
&\dots \\
V_0(x) &= \max_{a \in D_0} x\left\{E\left[G_1\left(a_*^1\right)Y_a^0 | Y_a^0 > 0\right] P\left(Y_a^0 > 0\right) + E\left[Y_a^0 | Y_a^0 \leqslant 0\right] P\left(Y_a^0 \leqslant 0\right)\right\} \\
&= \max_{a \in D_0} xG_0(a) = xG_0\left(a_*^0\right), \text{ where } x > 0.
\end{aligned}
$$

Here D_t, the set of admissible portfolios at stage t, is (see (5))

$$
D_t = \left\{a \in R^n : \sqrt{aC_ta'} \leqslant \frac{\langle a, \Delta m^t \rangle + m_0^t + d^t}{\frac{\varphi(\Phi^{-1}(\alpha^t))}{\alpha^t}}\right\}, \tag{6}
$$

where, recall, $\Delta m^t = (m_1^t - m_0^t, \dots, m_n^t - m_0^t)$, $t = 0, \dots, T-1$, random variables Y_a^t are defined above. Note that D_t does not depend on the current state $x > 0$ of the process X_t.

The functions $G_t(a)$ introduced above are defined by the recurrence formula

$$
\begin{aligned}
G_t(a) &= E\left[Y_a^t | Y_a^t > 0\right] P\left(Y_a^t > 0\right) G_{t+1}\left(a_*^{t+1}\right) \\
&\quad + E\left[Y_a^t | Y_a^t \leqslant 0\right] P\left(Y_a^t \leqslant 0\right), \ t = 0, \dots, T-1, \\
G_T(a) &\equiv 1,
\end{aligned} \tag{7}
$$

where a_*^{t+1} is the portfolio that maximizes $G_{t+1}(a)$ on the set D_{t+1}.

From the given expressions for the Bellman functions it follows that each portfolio a_*^t in the optimal strategy $A_* = (a_*^0, \dots, a_*^{T-1})$ depends only on the moment t of decision making and does not depend on the value of the current state $x > 0$ of the process X_t.

The following theorem gives an explicit expression for the functions $G_t(a)$ and provides the necessary optimality conditions in the optimal control problem (5).

Theorem 1. *Let the condition* (3) *be satisfied for all* α^t, σ_j^t, Δm_j^t.

If the investment strategy $(a_*^0, \dots, a_*^{T-1})$ *is optimal, then there exists a vector*

$$
(\lambda_1, \dots, \lambda_n, \mu) \in K^* = \left\{(\lambda, \mu) \in R^{n+1} : \sqrt{\lambda C_t^{-1} \lambda'} \leqslant \mu\right\},
$$

such that:

$$
\nabla G_t\left(a_*^t\right) + \frac{\Delta m^t \mu}{\frac{\varphi(\Phi^{-1}(\alpha^t))}{\alpha^t}} + \lambda = 0 \tag{8}
$$

and

$$a_*^t \lambda' + \frac{\mu\left(\langle a_*^t, \Delta m^t \rangle + m_0^t + d^t\right)}{\frac{\varphi(\Phi^{-1}(\alpha^t))}{\alpha^t}} = 0, \tag{9}$$

where $\nabla G_t\,(a)$ *denote gradient* $G_t\,(a)$*,*

$$\begin{aligned} G_t\,(a) = &\left\{\overline{\Phi}\left[-\frac{\mu^t\,(a)}{\sigma^t\,(a)}\right]\mu^t\,(a) + \sigma^t\,(a)\,\varphi\left[-\frac{\mu^t\,(a)}{\sigma^t\,(a)}\right]\right\} \\ &\times G_{t+1}\left(a_*^{t+1}\right) + \Phi\left[-\frac{\mu^t\,(a)}{\sigma^t\,(a)}\right]\mu^t\,(a) - \sigma^t\,(a)\,\varphi\left[-\frac{\mu^t\,(a)}{\sigma^t\,(a)}\right], \\ G_T(a) \equiv 1.& \end{aligned} \tag{10}$$

Here $\overline{\Phi}\,(\cdot) = 1 - \Phi\,(\cdot)$, $\Phi(\cdot)$ *and* $\varphi(\cdot)$ *denote the distribution function and density of the standard normal random variable, respectively,* $\mu^t\,(a) = \langle a, \Delta m^t \rangle + m_0^t$ *and* $\sigma^t(a) = \sqrt{aC_t a'}$.

Proof. The expression for $G_t\,(a)$ (see the recurrence formula (7)) is easily transformed into (10) taking into account the expressions for the mathematical expectation of the normal random variable Y_a^t, truncated to the intervals $(0, \infty)$ and $(-\infty, 0]$ (see, for example, [16]). Since the denominators in (10) contain the standard deviation $\sigma^t\,(a) = \sqrt{aC_t a'}$, it is necessary to first show that the degenerate investment portfolio $a^d = (0, \ldots, 0)$ cannot be optimal in the problem

$$\max_{a \in D_t} G_t\,(a)\,. \tag{11}$$

Indeed, consider the portfolio $a_\gamma = \gamma \Delta m^t$, which is admissible for sufficiently small $\gamma > 0$ (see (6)). Next,

$$\mu^t\,(a_\gamma) = \gamma \|\Delta m^t\|^2 + m_0^t, \quad \sigma^t\,(a_\gamma) = \gamma\sqrt{\Delta m_t C_t \Delta m_t'} \text{ and } \Phi\left(\frac{-\mu^t\,(a_\gamma)}{\mu^t\,(a_\gamma)}\right) \to 0,$$

$$\varphi\left(\frac{-\mu^t\,(a_\gamma)}{\sigma^t\,(a_\gamma)}\right) \to 0 \text{ when } \gamma \to 0 + 0.$$

Then

$$G_t\,(a_\gamma) = G_t\left(a_*^{t+1}\right)\left(\gamma\|\Delta m_t\|^2 + m_0^t\right) + o(\gamma) > G_t\left(a^d\right) = G_t\left(a_*^{t+1}\right) m_0^t$$

for sufficiently small values of $\gamma > 0$. Thus, to solve the problem (11) it is sufficient to limit ourselves to the set of admissible portfolios $D_t \setminus a^d$.

Condition (3) means that $IntD_t \neq \emptyset$, i.e., Slater's condition in problem (11) is satisfied. Applying Statement 2 to this cone programming problem, where the objective function $G_t\,(a)$ is not, in general, concave, we obtain (8), (9) as necessary conditions for optimality in (11). Theorem 1 is proved.

4. NON-NORMAL RISK ASSET RETURN MODEL

Let the returns of n risky assets have a multivariate elliptical distribution (normal distribution, Laplace distribution, Bessel distribution, etc. [17]), which allows, in particular, to take into account the "heavy tails" in the return distributions. A convenient property of this class of distributions for risk analysis is that a linear combination of elliptically distributed assets again has an elliptical distribution. Let $F(x)$ denote the distribution function of a "standard" elliptical random variable with zero mean and unit variance, and let $f(x)$ denote its density.

Then (see Section 2) the risk measure CVaR has the form

$$\mathrm{CVaR}_\alpha(X_a) = -EX_a + \sqrt{DX_a}k^*,$$

where now

$$k^* = \frac{-\int\limits_{-\infty}^{-z^*_\alpha} xf(x)dx}{\alpha} \quad \text{and} \quad z^*_\alpha : F(-z^*_\alpha) = \alpha.$$

Thus, the results obtained above remain valid in this case with $k = \phi\left(\Phi^{-1}(\alpha)\right)/\alpha$ replaced by k^*. Truncated normal distributions are replaced by corresponding truncated elliptical distributions.

5. EXAMPLE[1]

Let us illustrate the results obtained in Section 3 by solving a numerical example of finding optimal portfolios for a market of three companies: Apple, Microsoft, Facebook. Data on the stock price for the period 05/07/2022–05/07/2023 (one year) were taken in open access from the Nasdaq exchange website (https://www.nasdaq.com/market-activity/stocks/). The obtained realizations of the returns of these assets approximately follow normal distributions (but in this work, a strict test of hypotheses about the normality of the distributions of returns by methods of mathematical statistics remains out of consideration), therefore the assumption about the normal distribution of the investor's total capital at each stage seems justified. Based on these empirical data, estimates of the vector of mathematical expectations $m = (m^t_0, \ldots, m^t_3)$ and the covariance matrix $C = C_t$ of the three risky assets are constructed. It is assumed that the vectors of mathematical expectations and covariance matrices do not depend on the stage number over the entire time interval $[0, T] = [0, 4]$, and the risk-free asset has a yield of $m_0 = 1$, i.e., investments in this asset are preserved and do not bring losses/profits. The initial capital of the investor $x_0 = 1$; the significance level $\alpha = 0.05$; the constant d, which limits the risk measure CVaR defined above, is a variable parameter.

According to the results of Section 3 (see Theorem 1), the optimal portfolio at stage t is defined as the solution to the problem

$$\max_{D_t} G_t(a), \quad t = 3, 2, 1, 0,$$

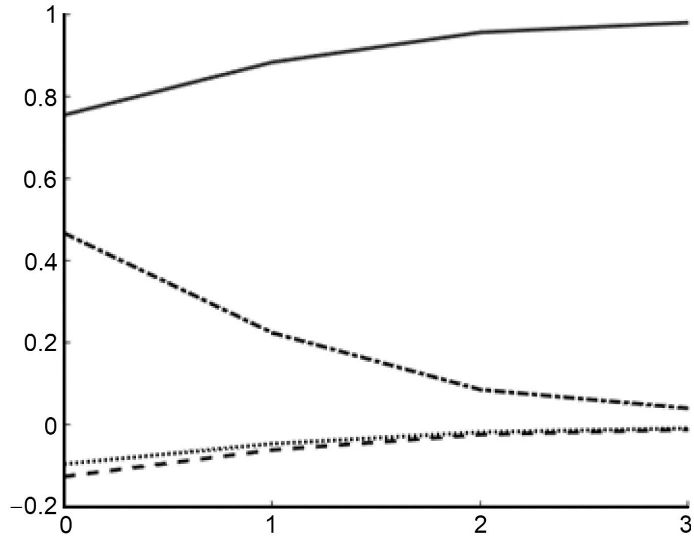
where the recurrence expressions for $G_t(a)$ are defined in (10). The table below shows how the optimal portfolio changes as the constant d, which constrains CVaR, increases.

Optimal investment portfolios in a multi-stage problem with a change in the constant d, which limits the risk measure CVaR at each stage

t	$t = 0$	$t = 1$	$t = 2$	$t = 3$
d	0.2	0.25	0.28	0.29
Apple, $a^t_*(1)$	0.7536	0.8820	0.9548	0.9786
Microsoft, $a^t_*(2)$	−0.0950	−0.0457	−0.0175	−0.0082
Facebook, $a^t_*(3)$	−0.1253	−0.0607	−0.0233	−0.0110
Risk-free asset, $a^t_*(0)$	0.4667	0.2244	0.0860	0.0406

With the increase of the constant d at each stage, the investor becomes more inclined to risk, so the share of investment in a risk-free asset is expected to decrease from stage to stage, and the share of investment in risky assets increases. For illustration, the obtained optimal investment portfolios are shown in figure.

[1] The data for this example and the solution itself were obtained with the help of Silventoinen D.P.



Optimal investment portfolios at stages $\{0, 1, 2, 3\}$. The solid line denotes the share of investments in Apple shares, the share of investments in the risk-free asset — dash-dotted line, the share of investments in Microsoft — dotted line, the share of investments in Facebook — dashed line.

The optimal value of the objective functional $J[A_*] = E\,X_4$ is equal to 1.031, which means obtaining 3% of the average profit for the entire investment period. This low value is explained by the fact that the observed annual returns on risky asset stocks are close to one and have low volatility — the differences in the empirical returns of each risky asset are only fractions of a percent. Note that the solved example is only an illustration of Theorem 1, and the methodology for calculating the optimal investment strategy given in it would give different results in the case of risky assets other than the three considered above.

6. CONCLUSION

The paper formulates and solves a multi-stage investment problem with probabilistic (CVaR) constraints and the possibility of bankruptcy. It is assumed that short selling is allowed and that the distribution of total returns at each stage is normal (or elliptical). It is shown that the optimal investment portfolio at each stage does not depend on the value of the investor's capital, but depends only on the number of the investment stage. In the context of finding optimal investment portfolios, it is shown that a multi-stage problem is reduced to a finite number of one-stage optimization problems that are recursively connected and are conical programming problems. For the auxiliary one-stage problem, sufficient conditions for the non-emptiness of the set of admissible portfolios and the fulfillment of the Slater regularity conditions are presented and proven. In solving the one stage problem, a modified Kuhn–Tucker theorem is applied for the case of generalized inequalities. The result of solving a multi-stage problem can be considered a method for constructing optimal investment portfolios, applicable in practice with known estimates of the vectors of mathematical expectations and matrices of covariances of returns. A numerical example based on one year of stock exchange price data for three companies is also presented to illustrate the theoretical results. The following directions of development of this work seem interesting: using other distributions instead of the normal or elliptical model for returns, for example, gamma distributions; using another risk measure instead of CVaR, for example, shortfall probability [3]. Another direction could be the analysis of a similar problem, but without of short sales.

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